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# Correlation Inequalities for Non-Purely-Ferromagnetic Systems

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Correlation inequalities are proven for spin systems with non-purely-ferromagnetic interactions possessing a certain symmetry. These inequalities generalize well-known inequalities of Griffiths, Ginibre, Lebowitz, Schrader, Messager–Miracle-Sole, and Percus.

**KEY WORDS:** Correlation inequalities; non-purely ferromagnetic systems.

## 1. INTRODUCTION

In classical statistical mechanics correlation inequalities can be successfully applied to study questions related to translation invariance.[1,2] The conjectured nonexistence of nontranslation-invariant states in the classical $Z^2$ Ising model could be a consequence of a correlation inequality which has been proven for a class of boundary conditions in Ref. 2. By a simple transformation this inequality can be written in a more symmetric form at the cost of introducing nonferromagnetic interactions. However, under certain symmetry conditions we are able to derive correlation inequalities for interactions that are not purely ferromagnetic and it is hoped that they give insight into nontranslation-invariant states. For example, they give a new proof of the results of Ref. 2, although they are unable to broaden the aforementioned class of boundary configurations.

Our inequalities are formulated in Section 2 and proven in Section 4. The motivation for this work is briefly described in Section 3.

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## 2. THE INEQUALITIES

Let $\Lambda$, $\Lambda^*$ be disjoint finite sets and $\varphi\colon \Lambda \to \Lambda^*$ a bijection. Let $F_0(\Lambda) = \{e\colon e \text{ is a function } \Lambda \to Z^+\}$, i.e., $F_0(\Lambda)$ contains all "subsets" of $\Lambda$ with possibly repeated elements. For $e \in F_0(\Lambda)$ denote $|e| = \sum_{i\in\Lambda} e_i$. Throughout the paper the ranges of the variables $i, j,\ldots$ and $K, A, B,\ldots$ will be $\Lambda$ and $F_0(\Lambda)$, respectively, and the same variables will also denote the images of these objects under $\varphi$.

On the set $M = \Lambda \cup \Lambda^*$ we consider the continuous spin variables $\sigma_i$, $\sigma_i^* \in R^1$, where $\sigma_i$ and $\sigma_i^*$ are the spins at the sites $i$ and $\varphi(i)$.

Let $\nu_i$ be symmetric measures on $L = [-a, a]$ $(0 < a \leqslant \infty)$, i.e., $d\nu_i(x) = d\nu_i(-x)$ $(i \in \Lambda)$. If $U(\sigma \times \sigma^*)$ is the energy of the configuration $\sigma \times \sigma^* = \{\sigma_i\} \times \{\sigma_j\}$ and $f\colon \sigma \times \sigma^* \to R^1$ is arbitrary, then $\langle f\rangle$ denotes the expectation value

$$\langle f\rangle = Z^{-1}\int_{L^{2|\Lambda|}} f(\sigma \times \sigma^*) \exp[-U(\sigma \times \sigma^*)] \prod_{i\in\Lambda} d\nu_i(\sigma_i)\, d\nu_i(\sigma_i^*)$$

For $a < \infty$ denote by $\mathbf{C}(L^{|\Lambda|})$ the algebra of real, continuous functions on $L^{|\Lambda|}$ supplied by the supremum norm. For $S \subset \mathbf{C}(L^{|\Lambda|})$ let $Q(S)$ be the closed, convex, multiplicative cone spanned by $S$ (cf. Ref. 3) and fix $S = \{\sigma_i\colon i \in \Lambda\}$. In case $a = \infty$, $Q_0(S)$ will denote the convex multiplicative cone spanned by $S$ in the (nonnormed) algebra of measurable functions $R^{|\Lambda|} \to R^1$.

We will consider polynomial interactions of the form

$$-U(\sigma \times \sigma^*) = \sum_{K\in F_0(\Lambda)} (J_K\sigma^K + J_K{}^*\sigma^{*K} + L_K\sigma^K\sigma^{*K}) \tag{1}$$

where the sum is a finite one and $\sigma^K = \prod_{i\in K} \sigma_i^{e_i}$.

If $a = \infty$, then to ensure the convergence of the integrals we suppose that the tails of $\nu_i$ decay sufficiently rapidly, that is, if $d$ is the degree of the polynomial (1), then (see Ref. 11)

$$\int_{R^1} \exp(b|s|^d)\, d\nu_i(s) < \infty \qquad \forall b \in R^1 \tag{2}$$

**Theorem 1.** If $J_K \geqslant |J_K{}^*|$ and $L_K$ is arbitrary, then

$$\left\langle \prod_{k=1}^{N} [f_k(\sigma) \pm f_k(\sigma^*)] \right\rangle \geqslant 0$$

for any $N \geqslant 1, f_k \in Q_0(S)$ [also for $f_k \in Q(S)$ if $a < \infty$], and arbitrary choice of signs.

In the case $a = \infty$, the validity of Theorem 1 can of course be extended to functions $f_k$ in the closure of $Q_0(S)$ in a suitable chosen $L_p$ space.

Introduce the "duplicate" variables:

$$t_i = 2^{-1/2}(\sigma_i + \sigma_i^*) \qquad \text{and} \qquad q_i = 2^{-1/2}(\sigma_i - \sigma_i^*) \tag{3}$$

implying that $\sigma_i = 2^{-1/2}(t_i + q_i)$ and $\sigma_i^* = 2^{-1/2}(t_i - q_i)$.

**Corollary.** If $0 < a \leqslant \infty$, then for any $A, B \in F_0(\Lambda)$:

(i) $\langle t^A q^B \rangle \geqslant 0$.
(ii) $\langle \sigma^A \rangle \geqslant 0$.
(iii) $\langle \sigma^A \sigma^B \rangle \geqslant \langle \sigma^A \sigma^{*B} \rangle$.
(iv) $\langle \sigma^A \sigma^B \rangle - \langle \sigma^{*A} \sigma^{*B} \rangle \geqslant |\langle \sigma^A \sigma^{*B} \rangle - \langle \sigma^{*A} \sigma^B \rangle| \geqslant 0$.

These statements generalize the Ginibre inequality,[3] the first GKS inequality,[4,5] the Schrader–Messager–Miracle-Sole inequality,[2,6] and the new Lebowitz inequality,[7,8] which implies an inequality of Griffiths.[9] A generalization of the Percus inequality[10] can also be given.

**Theorem 2.** If $0 \leqslant a < \infty$ and the interaction is given by

$$-U(\sigma \times \sigma^*) = \sum_{i \neq j} J_{ij}(\sigma_i \sigma_j + \sigma_i^* \sigma_j^*) + \sum_i (h_i \sigma_i + h_i^* \sigma_i^*) + \sum_K L_K \sigma^K \sigma^{*K}$$

with $J_{ij} \geqslant 0$, $h_i \geqslant h_i^*$, and the last sum is a finite one, then for any $A \in F_0(\Lambda)$

$$\langle q^A \rangle \geqslant 0$$

Finally, we formulate a completely new inequality.

**Theorem 3.** If $\nu_i = 1/2\delta_{-1} + 1/2\delta_1$ and the energy is given by

$$-U(\sigma \times \sigma^*) = \sum J_{ij}(\sigma_i \sigma_j + \sigma_i^* \sigma_j^*) + \sum L_i \sigma_i \sigma_i^* + \sum (h_i \sigma_i + h_i^* \sigma_i^*)$$

with $J_{ij} \geqslant 0$, $L_i$ arbitrary, and we consider two sets of external fields $h = \{h_i, h_i^*: i \in \Lambda\}$ and $\tilde{h} = \{\tilde{h}_i, \tilde{h}_i^*: i \in \Lambda\}$ satisfying the inequalities

$$\tilde{h}_i - \tilde{h}_i^* \geqslant |h_i - h_i^*|, \qquad |\tilde{h}_i + \tilde{h}_i^*| \leqslant h_i + h_i^*, \qquad \forall i \in \Lambda$$

then for any $A, B \in F_0(\Lambda)$

$$\langle t^A \rangle^{\tilde{h}} / \langle t^A \rangle^h \leqslant \langle q^B \rangle^{\tilde{h}} / \langle q^B \rangle^h$$

## 3. CONNECTION WITH TRANSLATION INVARIANCE

Let $\Lambda$ be a finite subset of $Z^d$ symmetric with respect to the line $x^1 = -1/2$. In Ref. 2 the inequality

$$\langle (\sigma_i + \sigma_{\bar{i}})(\sigma_j + \sigma_{\bar{j}}) \rangle^{\Lambda, b_\Lambda} \geqslant \langle (\sigma_i + \sigma_{\bar{i}})(\sigma_j + \sigma_{\bar{j}}) \rangle^{\Lambda, \pm} \tag{4}$$

has been conjectured for the classical spin-$\frac{1}{2}$ Ising model ($i, j \in \Lambda$; $i^1, j^1 \geqslant 0$) with formal interaction $-U(\sigma) = \beta \sum_{|i-j|=1} \sigma_i \sigma_j$ ($\beta \geqslant 0$). In this inequality $\bar{\imath}$ denotes the site symmetric to $i$ with respect to the line $x^1 = -1/2$ and $\langle \cdots \rangle^{\Lambda, b_\Lambda}$ denotes the expectation with respect to the Gibbs distribution $P^{\Lambda, b_\Lambda}$ corresponding to an arbitrary boundary configuration $b_\Lambda$ (i.e., $b_\Lambda: \partial\Lambda \to \{-1, +1\}$, $\partial\Lambda = \{j: j \in Z^d - \Lambda, \exists\ i \in \Lambda$, such that $|i - j| = 1\}$, while $\pm$ denotes the boundary configurations equal to $\operatorname{sgn}(i^1 + 1/2)$ for $i = (i^1, \ldots, i^d) \in \partial\Lambda$.

For $d = 2$ it has been shown by Gallavotti[12] and Abraham and Reed[13] that the infinite Gibbs state $P^{\pm} = \lim_{\Lambda \to Z^2} P^{\Lambda, \pm}$ is translation invariant, namely $P^{\pm} = 1/2P^{-} + 1/2P^{+}$. This fact and the inequality (4) would imply the famous conjecture that $G(\beta) = G^{\mathrm{inv}}(\beta)$ for every $\beta \geqslant 0$, i.e., no non-translation-invariant states exist ($d = 2$!).

In Ref. 2, (4) has been proven for the class of boundary configurations satisfying $b_i + b_{\bar{\imath}} \geqslant 0$ for every $i \in \partial\Lambda$ with $i^1 \geqslant 0$ and this result has enabled the authors to prove the translation invariance of a certain class of infinite Gibbs states. It is easy to see that the same result is a consequence of our Theorem 3. We remark that from Theorem 3 it also follows that for the boundary configurations described above

$$\langle \sigma_i - \sigma_{\bar{\imath}} \rangle^{\Lambda, b_\Lambda} \leqslant \langle \sigma_i - \sigma_{\bar{\imath}} \rangle^{\Lambda, \pm} \qquad (i \in \Lambda, \quad i^1 \geqslant 0) \tag{5}$$

This inequality, if it is true for arbitrary boundary configurations, would imply the translation invariance of the one-point correlation $\langle \sigma_i \rangle^P$ of any $P \in G(\beta)$.

By multiplying all the spins by $\operatorname{sgn}(i^1 + 1/2)$, the inequalities (5) and (4) go over into

$$\langle \sigma_i + \sigma_{\bar{\imath}} \rangle^{\Lambda, b_\Lambda} \leqslant \langle \sigma_i + \sigma_{\bar{\imath}} \rangle^{\Lambda, +}$$

$$\langle (\sigma_i - \sigma_{\bar{\imath}})(\sigma_j - \sigma_{\bar{\jmath}}) \rangle^{\Lambda, b_\Lambda} \geqslant \langle (\sigma_i - \sigma_{\bar{\imath}})(\sigma_j - \sigma_{\bar{\jmath}}) \rangle^{\Lambda, +} \tag{6}$$

being understood for the interaction

$$-U(\sigma) = \beta \sum_{\substack{|i-j|=1 \\ i^1, j^1 \geqslant 0}} (\sigma_i \sigma_j + \sigma_{\bar{\imath}} \sigma_{\bar{\jmath}}) - \beta \sum_{|i - \bar{\imath}|=1} \sigma_i \sigma_{\bar{\imath}} + \beta \sum_{\substack{|i-j|=1 \\ i^1, j^1 \geqslant 0}} (\sigma_i b_j + \sigma_{\bar{\imath}} b_{\bar{\jmath}})$$

We expect that the inequalities (4) and (5) can be more easily attacked in their more symmetric forms (6), though for the time being we do not have positive results.

## 4. PROOFS

We manifest the inequalities by the method of duplicate variables.[10,11]

### Proof of Theorem 1

**Lemma 1.** Suppose that $d\mu(t, q)$ is a measure on $R^{2s}$ with the symmetries $d\mu(t, q) = d\mu({}_kt, q) = d\mu(t, {}_kq)$ $(t, q \in R^s, 1 \leqslant k \leqslant s)$, where ${}_k(x_1,..., x_s) = (x_1,..., x_{k-1}, -x_k, x_{k+1},..., x_s)$. Then

$$\int_{R^{2s}} \prod_{l=1}^{s} (t_l^{m_l} q_l^{n_l})\, d\mu(t, q) \geqslant 0 \qquad (m_l, n_l \in Z^+)$$

*Proof of Lemma 1.* If any of the $m_l$ or $n_l$ is odd, then the integral vanishes by the symmetries of $\mu$. Otherwise the integrand is nonnegative.

It is sufficient to prove (2) for $f_k \in S$ $(1 \leqslant k \leqslant N)$. In this case (2) reduces to inequality (i) of the Corollary. Its LHS is

$$\langle t^A q^B \rangle = \int_{R^{2|\Lambda|}} t^A q^B \exp\left[\sum_K J_K \left(\frac{t+q}{\sqrt{2}}\right)^K + J_K{}^* \left(\frac{t-q}{\sqrt{2}}\right)^K\right]$$

$$\times \exp\left[\sum_K L_K \left(\frac{t^2 - q^2}{2}\right)^K\right] \prod_{i \in \Lambda} d\rho_i(t_i, q_i)$$

where

$$d\rho_i(t_i, q_i) = d\nu_i\left(\frac{t_i + q_i}{\sqrt{2}}\right) d\nu_i\left(\frac{t_i - q_i}{\sqrt{2}}\right)$$

is a measure on $R^2$. The first exponent can be expanded in a power series with positive coefficients and the measure

$$d\mu(t, q) = \exp\left[\sum_K L_K \left(\frac{t^2 - q^2}{2}\right)^K\right] \prod_{i \in \Lambda} d\rho_i(t_i, q_i)$$

possesses the symmetries required by the previous lemma.

Statements (ii)–(iv) of the Corollary follow from

$$\sigma^A = \left(\frac{t+q}{\sqrt{2}}\right)^A$$

$$\sigma^A(\sigma^B - \sigma^{*B}) = \left(\frac{t+q}{\sqrt{2}}\right)^A \left[\left(\frac{t+q}{\sqrt{2}}\right)^B - \left(\frac{t-q}{\sqrt{2}}\right)^B\right]$$

$$(\sigma^A \pm \sigma^{*A})(\sigma^B \mp \sigma^{*B}) = \left[\left(\frac{t+q}{\sqrt{2}}\right)^A \pm \left(\frac{t-q}{\sqrt{2}}\right)^A\right]\left[\left(\frac{t+q}{\sqrt{2}}\right)^B \mp \left(\frac{t-q}{\sqrt{2}}\right)^B\right]$$

## Proof of Theorem 2

The energy expressed in the new variables is

$$-\left\{\sum_{i\neq j} J_{ij}(t_i t_j + q_i q_j) + 2^{-1/2}\sum_i [(h_i + h_i^*)t_i + (h_i - h_i^*)q_i] + \sum_K L_K\left(\frac{t^2 - q^2}{2}\right)^K\right\}$$

When calculating $\langle q^A \rangle$, the factor

$$\exp\left\{\sum_{i\neq j} J_{ij}q_i q_j + 2^{-1/2}\sum_i (h_i - h_i^*)q_i\right\}$$

can be expanded in a power series with positive coefficients; thus it is sufficient to show that

$$\int q^B \exp\left[\sum_{i\neq j} J_{ij}t_i t_j + 2^{-1/2}\sum_i (h_i + h_i^*)t_i + \sum_K L_K\left(\frac{t^2 - q^2}{2}\right)^K\right]\prod_{i\in\Lambda} d\rho_i(t_i, q_i) = \int q^B\, d\mu(t, q)$$

is nonnegative for any $t_i$: $i \in \Lambda$. This can be seen, however, by using the symmetry $d\mu(t, q) = d\mu(t, {}_K q)$ analogously as in Lemma 1.

## Proof of Theorem 3

The result can be obtained by a second duplication, introducing another copy of $\Lambda \cup \Lambda^*$, which will be denoted by $\tilde{\Lambda} \cup \tilde{\Lambda}^*$. Thus to any variable on $\Lambda \cup \Lambda^*$ there corresponds a unique variable on $\tilde{\Lambda} \cup \tilde{\Lambda}^*$ that will be designated by a tilde. For any $i \in \Lambda$ we introduce the variables $\alpha_i = 2^{-1/2}(t_i + \tilde{t}_i)$, $\beta_i = 2^{-1/2}(t_i - \tilde{t}_i)$, $\gamma_i = 2^{-1/2}(q_i + \tilde{q}_i)$, and $\delta_i = 2^{-1/2}(\tilde{q}_i - q_i)$. Let the energy on $\Lambda \cup \Lambda^* \cup \tilde{\Lambda} \cup \tilde{\Lambda}^*$ be

$$-\left\{\sum J_{ij}(\sigma_i\sigma_j + \sigma_i^*\sigma_j^* + \tilde{\sigma}_i\tilde{\sigma}_j + \tilde{\sigma}_i^*\tilde{\sigma}_j^*) + \sum L_i(\sigma_i\sigma_i^* + \tilde{\sigma}_i\tilde{\sigma}_i^*) + \sum (h_i\sigma_i + h_i^*\sigma_i^* + \tilde{h}_i\tilde{\sigma}_i + \tilde{h}_i^*\tilde{\sigma}_i^*)\right\}$$

or in terms of the variables $\alpha$, $\beta$, $\gamma$, and $\delta$

$$-\Big\{\sum J_{ij}(\alpha_i\alpha_j + \beta_i\beta_j + \gamma_i\gamma_j + \delta_i\delta_j) + (1/2)\sum L_i(\alpha_i^2 + \beta_i^2 - \gamma_i^2 + \delta_i^2) + (1/2)\sum [\alpha_i(h_i + h_i^* + \tilde{h}_i + \tilde{h}_i^*) + \beta_i(h_i + h_i^* - \tilde{h}_i - \tilde{h}_i^*) + \gamma_i(h_i - h_i^* + \tilde{h}_i - \tilde{h}_i^*) + \delta_i(-h_i + h_i^* + \tilde{h}_i - \tilde{h}_i^*)]\Big\}$$

Since

$$\alpha_i^2 + \beta_i^2 - \gamma_i^2 - \delta_i^2 = 4 - 2(\gamma_i^2 + \delta_i^2) = 2(\alpha_i^2 + \beta_i^2) - 4$$

we can always find such a representation of the sum in brackets where, except for the constant term, all the coefficients are nonnegative.

The theorem states that in this new system

$$\langle t^A \tilde{q}^B - t^A \tilde{q}^B \rangle \geqslant 0$$

i.e.,

$$\left\langle \left(\frac{\alpha + \beta}{\sqrt{2}}\right)^A \left(\frac{\gamma + \delta}{\sqrt{2}}\right)^B - \left(\frac{\alpha - \beta}{\sqrt{2}}\right)^A \left(\frac{\gamma - \delta}{\sqrt{2}}\right)^B \right\rangle \geqslant 0$$

This inequality, however, comes from an Ellis–Monroe type inequality $\langle \alpha^A \beta^B \gamma^C \delta^D \rangle \geqslant 0$ that can be proven for our potential in the usual way.[11]

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